FINAL MARK

GIRRAWEEN HIGH SCHOOL
MATHEMATICS EXTENSION 2
2023
HSC TASK 3
ANSWERS COVER SHEET

Student Number: _____

QUESTION	MARK	MEX12-1	MEX12-2	MEX12-3	MEX12-4	MEX12-5	MEX12-6	MEX12-7	MEX12-8
1-5	/5			✓		✓		✓	✓
6	/12			✓				✓	✓
7	/11			✓				✓	✓
8	/9			✓				✓	✓
9	/14					✓		✓	✓
10	/17					✓		✓	✓
11	12					✓		✓	✓
TOTAL									
	/80			/37		/48		/80	/80

Year 12 Mathematics Extension 2 outcomes

A student:

MEX12-1 understands and uses different representations of numbers and functions to model, prove results and find solutions to problems in a variety of contexts

MEX12-2 chooses appropriate strategies to construct arguments and proofs in both practical and abstract settings

MEX12-3 uses vectors to model and solve problems in two and three dimensions

MEX12-4 uses the relationship between algebraic and geometric representations of complex numbers and complex number techniques to prove results, model and solve problems

MEX12-5 applies techniques of integration to structured and unstructured problems

MEX12-6 uses mechanics to model and solve practical problems

MEX12-7 applies various mathematical techniques and concepts to model and solve structured, unstructured and multi-step problems

MEX12-8 communicates and justifies abstract ideas and relationships using appropriate language, notation and logical argument

Select the alternative A, B, C or D that best answers the question.

1.	A ⬭	B. ⬭	C. ⬭	D. ⬭
2.	A ⬭	B. ⬭	C. ⬭	D. ⬭
3.	A ⬭	B. ⬭	C. ⬭	D. ⬭
4.	A ⬭	B. ⬭	C. ⬭	D. ⬭
5.	A ⬭	B. ⬭	C. ⬭	D. ⬭

GIRRAWEEN HIGH SCHOOL
MATHEMATICS
Extension 2
2023
Year 12 Task 3

Time Allowed: 90 minutes

Instructions:

- There are 11 questions in this paper. All questions are compulsory.
- Use blue or black pen.
- Write all your answers in the Answer Booklets provided.
- For Questions 1 – 5, fill in the circle corresponding to the correct answer in your answer booklet.
- For Questions 6 – 11, start each question on a new page.
- Write on both sides of the paper.
- Show all necessary working.
- Board-approved calculators may be used.
- Mathematics reference sheets are provided.
- Marks may be deducted for careless or badly arranged work.

Questions 1 - 5 (5 marks)

Colour in the circle corresponding to the correct answer on the answer sheet provided in your answer booklet.

1. What is the unit vector in the same direction as $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} - 3\underset{\sim}{k}$?

   A. $\frac{1}{14}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

   B. $\frac{-1}{14}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

   C. $\frac{-1}{\sqrt{14}}\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$

   D. $\frac{1}{\sqrt{14}}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$

2. What is the size of the angle between the vectors $3\underset{\sim}{i} + 6\underset{\sim}{j} - 2\underset{\sim}{k}$ and $2\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$?

   A. $112°24'$ B. $100°59'$ C. $79°01'$ D. $67°36'$

3. The vector equation of a line joining the points $L(1, 3, 5)$ and $M(x, y, z)$ is

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix}$$

   Which of the following could be the coordinates of M ?

   A. (1, 5, - 4) B. (3, 11, 8) C. (3, 12, 6) D. (3, 11, 6)

4. Which of the following expressions is equal to $\int \frac{1}{x^2 + 6x + 14}\, dx$?

A. $\frac{1}{2\sqrt{5}} \ln \frac{(x + 3) - \sqrt{5}}{(x + 3) + \sqrt{5}} + c$

B. $\ln \frac{(x + 3) - \sqrt{5}}{(x + 3) + \sqrt{5}} + c$

C. $\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{x + 3}{\sqrt{5}}\right) + c$

D. $\tan^{-1}\left(\frac{x + 3}{\sqrt{5}}\right) + c$

5. Which of the following integrals is equivalent to $\int \sqrt{\frac{1 + x}{1 - x}}\, dx$?

A. $\int \sqrt{1 - x^2}\, dx$

B. $\int \frac{1 + x}{\sqrt{1 - x^2}}\, dx$

C. $\int \frac{1 + x}{1 - x}\, dx$

D. $\int \left(1 + \frac{2x}{\sqrt{1 - x}}\right) dx$

Question 6 (12 marks) [*Start a new page in your answer booklet*]

a. If $\overrightarrow{OA} = -\underset{\sim}{i} + 4\underset{\sim}{k}$ and $\overrightarrow{OB} = 4\underset{\sim}{i} + \underset{\sim}{j} + 3\underset{\sim}{k}$,

(i) Find $\overrightarrow{AB}$. [1]

(ii) Find $\left|\overrightarrow{AB}\right|$. [1]

(iii) Find the coordinates of point C such that $\overrightarrow{AC} = -\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$. [2]

b. Determine the vector equation, $\underset{\sim}{r}$, of the line segment *AB* where $A = (1, 2, 1)$ and $B = (4, 10, -3)$. [2]

c. Find the coordinates of the point that divides A (2, - 3, 10) and B (- 3, 12, 0) internally in the ratio 2:3. [3]

d. Consider the vectors $\underset{\sim}{u} = \underset{\sim}{i} + a\underset{\sim}{j} - \underset{\sim}{k}$ and $\underset{\sim}{v} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$.

For what values of a are $\underset{\sim}{u} + \underset{\sim}{v}$ and $\underset{\sim}{u} - \underset{\sim}{v}$ perpendicular? [3]

Question 7 (11 marks) [*Start a new page in your answer booklet*]

a. Consider the vectors $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ and $\overrightarrow{OC} = \begin{pmatrix} 3 \\ \mu \\ \lambda \end{pmatrix}$,

where *0* is the origin.

Find the values of μ and λ if the points A, B and C are collinear. [3]

b. The Cartesian equation of a sphere is $x^2 + y^2 + z^2 - 4x + 10z + 20 = 0$.

Find the vector equation of the sphere. [3]

c. Three points $A(-2, 1, 7)$, $B(-1, 3, 5)$ and $C(-4, 7, 3)$ lie in three-dimensional space. The points A and B lie on the line l.

(i) Express $\overrightarrow{AB}$ and $\overrightarrow{AC}$ as column vectors. [1]

(ii) Find the shortest distance of C from the line l. [4]

Question 8 (9 marks) [*Start a new page in your answer booklet*]

a. Determine whether the following lines are parallel, intersecting or skew. Show all working. [3]

$$\underset{\sim}{r}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

$$\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

b. Consider the two spheres with equations given below:

$$\left| \underset{\sim}{r} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \right| = 5$$

$$\left| \underset{\sim}{r} - \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} \right| = 10$$

(i) Show that the spheres have a single point of contact. [3]

(ii) Find the coordinates of the point of contact. [3]

Question 9 (14 marks) [*Start a new page in your answer booklet*]

Find:

a. $\int x^2 e^{x^3}\ dx$ [2]

b. $\int \tan^3 x \sec^2 x\ dx$ [2]

c. $\int e^{3x}\sqrt{e^{3x}+1}\ dx$ [3]

d. $\int \dfrac{3x+2}{x^2+6x+10}\ dx$ [3]

e. $\int \cos^4 x\ dx$ [4]

Question 10 (17 marks) [*Start a new page in your answer booklet*]

a. Find $\int \dfrac{1}{1+2\sin x-\cos x}\ dx$ [4]

b. (i) Express $\dfrac{2x}{(x+1)(x^2+1)}$ in the form $\dfrac{A}{x+1}+\dfrac{Bx+C}{x^2+1}$ [3]

where A, B and C are real.

(ii) Hence, find $\displaystyle\int_0^1 \frac{2x}{(x+1)(x^2+1)}\ dx$ [4]

c. The integral I_n is defined by $I_n = \displaystyle\int_0^1 e^x\ (2-x)^n\ dx$ for integers $n \geq 0$.

(i) Show that $I_n = e - 2^n + nI_{n-1}$ [3]

(ii) Hence, evaluate $\displaystyle\int_0^1 e^x\ (2-x)^4\ dx$ [3]

Question 11 (12 marks) [*Start a new page in your answer booklet*]

a. Use the property $\int_0^a f(x)\,dx = \int_0^a f\,(a-x)\,dx$ to evaluate [4]

$$\int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\,dx$$

b. (i) If $I_n = \int_0^{\frac{\pi}{4}} \sec^n x\,dx$, show that $I_n = \frac{\left(\sqrt{2}\right)^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ [4]

(ii) Hence, find $\int_0^{\frac{\pi}{4}} \sec^6 x\,dx$ [4]

End of Examination

2023 GHS MATHEMATICS EXTENSION 2 TASK 3

SOLUTIONS

1. $\hat{\underset{\sim}{u}} = \dfrac{\underset{\sim}{u}}{|\underset{\sim}{u}|}$

$= \dfrac{2i + j - 3k}{\sqrt{14}}$

D

2. $\cos\theta = \dfrac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$

$= \dfrac{3\times 2 + 6\times -2 - 2\times 1}{\sqrt{49}\times\sqrt{9}}$

$= \dfrac{-8}{21}$

$\theta = 112^\circ 24'$

A

3. $\overrightarrow{LM} = \begin{pmatrix} x-1 \\ y-3 \\ z-5 \end{pmatrix}$, $\underset{\sim}{r} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ 8 \\ 1 \end{pmatrix}$

If $\lambda = 1$, $\underset{\sim}{r} = \begin{pmatrix} 3 \\ 11 \\ 6 \end{pmatrix}$

D

4. $\displaystyle\int \frac{1}{x^2+6x+14}\,dx = \int \frac{dx}{(x+3)^2 + (\sqrt{5})^2}$

$= \dfrac{1}{\sqrt{5}} \tan^{-1}\dfrac{(x+3)}{\sqrt{5}} + C$

C

5. $\sqrt{\dfrac{1+x}{1-x}} = \dfrac{1+x}{\sqrt{1-x}\sqrt{1-x}}$

$= \dfrac{1+x}{\sqrt{1-x^2}}$

B

Question 6 (12 marks)

a)

i) $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$

$= \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$

$= \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

$\overrightarrow{AB} = 5\underset{\sim}{i} + \underset{\sim}{j} - \underset{\sim}{k}$ (1)

ii) $|\overrightarrow{AB}| = \sqrt{5^2 + 1^2 + (-1)^2}$

$= \sqrt{27}$

$= 3\sqrt{3}$ (1)

well done

iii) $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$

$= \begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$\therefore C = (-2, 1, 3)$ (2)

well done

b) $A(1, 2, 1)$; $B = (4, 10, -3)$

$\underset{\sim}{b} - \underset{\sim}{a} = \begin{pmatrix} 4 \\ 10 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}$

$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}$

well done

But $\underset{\sim}{r}$ represents a point on line segment $\overline{AB}$.

When $\lambda = 0$, $\underset{\sim}{r} = \overrightarrow{OA}$. when $\lambda = 1$, $\underset{\sim}{r} = \overrightarrow{OB}$

$\therefore \underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 8 \\ -4 \end{pmatrix}$, $0 \leq \lambda \leq 1$ (2)

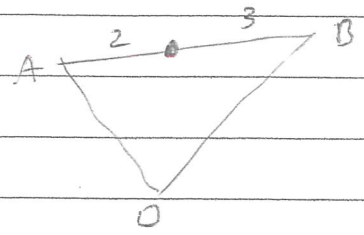
a lot of students did not write this

(2)

c) $A(2,-3,10)$; $B(-3,12,0)$

$$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$$

$$= \begin{pmatrix} -5 \\ 15 \\ -10 \end{pmatrix}$$



$$\underset{\sim}{r} = \begin{pmatrix} 3 \\ -3 \\ 10 \end{pmatrix} + \frac{2}{5}\begin{pmatrix} -5 \\ 15 \\ -10 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 3 \\ 6 \end{pmatrix}$$

- Generally well done.
- some errors with negative signs

(3)

$\therefore (0,3,6)$ divides $\overline{AB}$ in the ratio 2:3.

d) $\underset{\sim}{u} = \underset{\sim}{i} + a\underset{\sim}{j} - \underset{\sim}{k}$; $\underset{\sim}{v} = -\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$

$$\underset{\sim}{u} + \underset{\sim}{v} = (a+1)\underset{\sim}{j}$$

$$\underset{\sim}{u} - \underset{\sim}{v} = 2\underset{\sim}{i} + (a-1)\underset{\sim}{j} - 2\underset{\sim}{k}$$

$$(\underset{\sim}{u} + \underset{\sim}{v}) \cdot (\underset{\sim}{u} - \underset{\sim}{v}) = (a+1)(a-1)$$

$$= a^2 - 1$$

The two vectors are $\perp$ if

$$a^2 - 1 = 0$$

$$\therefore \quad a = \pm 1$$

Well done

(3)

Question 7 (11 marks)

a) $\overrightarrow{OA} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$; $\overrightarrow{OB} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$; $\overrightarrow{OC} = \begin{pmatrix} 3 \\ \mu \\ \lambda \end{pmatrix}$

$\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$; $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ $= \begin{pmatrix} 2 \\ \mu - 1 \\ \lambda \end{pmatrix}$

Collinear $\therefore \overrightarrow{AB} \parallel \overrightarrow{AC}$ $= 2\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\therefore \mu - 1 = -4$ and $\lambda = 2$

$\mu = -3$ (3)

b) $x^2 + y^2 + z^2 - 4x + 10z + 20 = 0$

$x^2 - 4x + y^2 + z^2 + 10z = -20$ $+4$ $+25$

$(x-2)^2 + y^2 + (z+5)^2 = 9$

Centre $(2, 0, -5)$; radius $= 3$

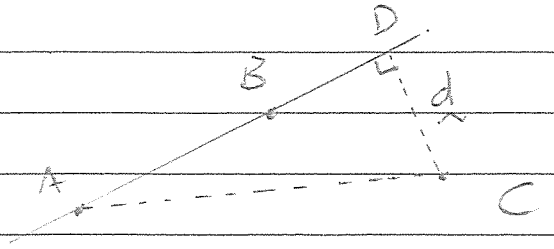
Vector equation : $\left| \underset{\sim}{r} - \begin{pmatrix} 2 \\ 0 \\ -5 \end{pmatrix} \right| = 3$ (3)

c) $A(-2, 1, 7)$; $B(-1, 3, 5)$; $C(-4, 7, 3)$

i) $\overrightarrow{AB} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$ (1)

Q.(7)(cont).

(ii)



$$\text{proj}_{\overrightarrow{AB}} \overrightarrow{AC} = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\overrightarrow{AB} \cdot \overrightarrow{AB}} \cdot \overrightarrow{AB}$$

$$= \frac{-2+12+8}{1+4+4} \cdot \overrightarrow{AB}$$

$$= 2\overrightarrow{AB}$$

$$= \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix}$$

$$\therefore \underset{\sim}{d} = \overrightarrow{AD} - \overrightarrow{AC}$$

$$= \begin{pmatrix} 2 \\ 4 \\ -4 \end{pmatrix} - \begin{pmatrix} -2 \\ 6 \\ -4 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$$

$$|\underset{\sim}{d}| = \sqrt{4^2 + (-2)^2}$$

$$= \sqrt{20}$$

$$= 2\sqrt{5} \text{ units}$$

∴ Shortest distance of C from l

$= 2\sqrt{5}$ units

(4)

Q.(7) Feedback:

(a) Some students forgot that A, B, C are collinear, $\vec{BC} = k\vec{AB}$ (not just $\vec{AB}$).

So $\begin{pmatrix} 1 \\ \mu+1 \\ \lambda-1 \end{pmatrix} = k\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ → (In this case $k=1$, but if students used $\vec{AC} = k\vec{AB}$ then $k=\frac{1}{2}$).

(b) Very well done.

(c) The simplest & most reliable way to do this was using a projection (see solutions). There were some weird & wonderful methods shown (see below) but they had a much higher error rate.

Method 1:

If D is closest point on AB to C, then $\vec{CD} \cdot \vec{AB} = 0$ as $CD \perp AD$.

So as $D = \begin{pmatrix} \lambda-2 \\ 2\lambda+1 \\ -2\lambda+7 \end{pmatrix}$

$\vec{CD} \cdot \vec{AB} = 0$

$\begin{pmatrix} \lambda+2 \\ 2\lambda-6 \\ -2\lambda+4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = 0$

$9\lambda - 18 = 0$

$\lambda = 2.$

$\vec{CD} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$

& perpendicular distance is

$= \sqrt{4^2+(-2)^2+0^2}$

$= 2\sqrt{5}.$

Method 2:

If D lies on AB then we find the MINIMUM value of $|CD|$

$|CD|^2$

$= (\lambda+2)^2 + (2\lambda-6)^2 + (-2\lambda+8)^2$

$= 9\lambda^2 - 36\lambda + 56$

Minimum value will occur at "turning point" (Quadratic).

(could also find $\frac{d(|CD|^2)}{d\lambda}$).

$|\vec{CD}|^2 = 9(\lambda-2)^2 + 20$

Minimum value of $|\vec{CD}|^2 = 20$

Minimum value of $|\vec{CD}| = 2\sqrt{5}$.

Question 8 (9 marks)

a) $\underset{\sim}{r}_1 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$; $\underset{\sim}{r}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

Test for //

$\frac{1}{1} = \frac{1}{1} \neq \frac{2}{3}$

$\therefore$ not parallel.

* Few didn't prove that the lines are not parallel.

Test for Intersecting

$\lambda = 1 + \mu$ — (1)

$2 + \lambda = \mu$ — (2)

* not writing the direction vectors

substituting (1) into (2)

$2 + 1 + \mu = \mu$

$3 = 0$ $\therefore$ no solution

$\therefore$ not intersection

$\therefore$ the lines are skew. (3)

b) i) $\left| \underset{\sim}{r} - \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} \right| = 5$; $\left| \underset{\sim}{r} - \begin{pmatrix} -7 \\ 7 \\ 1 \end{pmatrix} \right| = 10$

For a single point of contact the distance between centres must equal the sum of the radii.

Distance between Centres $= \left| \begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix} \right|$

$= \sqrt{100 + 121 + 4}$

$= \sqrt{225}$

$= 15$

$=$ sum of the radii

$\therefore$ the spheres have a single point of contact

done quite well

(6)

(3)

ii) Point of contact occurs at

$$\begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} + \frac{5}{15}\begin{pmatrix} -10 \\ 11 \\ -2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -4 \\ 3 \end{pmatrix} + \begin{pmatrix} \frac{-10}{3} \\ 11/3 \\ \frac{-2}{3} \end{pmatrix}$$

* students have done this in different ways.

∴ Point of contact

$$= \left(-\tfrac{1}{3}, -\tfrac{1}{3}, \tfrac{7}{3}\right)$$

(3)

Question 9 (14 marks)

a) $\int x^2 e^{x^3} dx$

$= \frac{1}{3} \int 3x^2 e^{x^3} dx$

$= \frac{1}{3} e^{x^3} + C$ (2)

Well done

b) $\int \tan^3 \sec^2 x \, dx$

$= \int \sec^2 x (\tan x)^3 dx$

$= \frac{\tan^4 x}{4} + C$ (2)

c) $\int e^{3x} \sqrt{e^{3x}+1} \, dx$

$u^2 = e^{3x} + 1$

$2u\,du = 3e^{3x} dx$

$e^{3x} dx = \frac{2u\,du}{3}$

$= \int \sqrt{u^2} \times \frac{2u\,du}{3}$

$= \frac{2}{3} \int u^2 du$

$= \frac{2}{3} \left(\frac{u^3}{3}\right) + C$

$= \frac{2\sqrt{(e^{3x}+1)^3}}{9} + C$ (3)

Getting $\frac{2}{3}$ instead of $\frac{2}{9}$

* Another correct solution for (b) is

$\frac{\sec^4 x}{4} - \frac{\sec^2 x}{2} + C$

d) $\int \frac{3x+2}{x^2+6x+10}\,dx$ ... Few students showed the correct step

$3x+2 = A(2x+6)+B$

$= \int \frac{\frac{3}{2}(2x+6)-7}{x^2+6x+10}\,dx$ $\quad 3x+2 = \frac{3}{2}(2x+6)-7$

$= \frac{3}{2}\int \frac{2x+6}{x^2+6x+10}\,dx - 7\int \frac{1}{(x+3)^2+1^2}\,dx$

$= \frac{3}{2}\ln(x^2+6x+10) - 7\tan^{-1}(x+3) + C$ (3)

e) $\int \cos^4 x\,dx$

$= \int (\cos^2 x)^2\,dx$

$= \int \left[\frac{1}{2}(1+\cos 2x)\right]^2\,dx$

$= \frac{1}{4}\int (1+2\cos 2x+\cos^2 2x)\,dx$

$= \frac{1}{4}\int \left(1+2\cos 2x+\frac{1}{2}(1+\cos 4x)\right)dx$

$= \frac{1}{4}\left[x+\sin 2x+\frac{1}{2}\left(\frac{1}{4}\sin 4x+x\right)\right]+C$

$= \frac{x}{4}+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}+\frac{x}{2}+C$

$= \frac{\sin 4x}{32}+\frac{\sin 2x}{4}+\frac{3x}{8}+C$

- Used wrong formula for $\cos^2 2x$
- not using brackets correctly. As a result making mistakes.

(4)

Question 10 (16 marks)

a) $\int \frac{1}{1+2\sin x - \cos x}\, dx$

let $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$= \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$

$= \frac{1}{2}(1+t^2)$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$= \int \frac{1}{1+2\left(\frac{2t}{1+t^2}\right) - \frac{1-t^2}{1+t^2}} \cdot \frac{2\,dt}{1+t^2}$

$= \int \frac{2}{1+t^2+4t-1+t^2}\, dt$

$= \int \frac{2}{2t^2+4t}\, dt$

$= \int \frac{1}{t^2+2t}\, dt$

$= \int \frac{1}{t(t+2)}\, dt$

$\frac{1}{t(t+2)} = \frac{A}{t} + \frac{B}{t+2}$

$1 = A(t+2) + Bt$

let $t=0$, $2A = 1$

$A = \frac{1}{2}$

when $t=-2$, $-2B = 1$

$B = -\frac{1}{2}$

$= \frac{1}{2}\int\left(\frac{1}{t} - \frac{1}{t+2}\right) dt$

$= \frac{1}{2}\left(\ln t - \ln(t+2)\right) + C$

$= \frac{1}{2}\left[\ln\left(\tan\frac{x}{2}\right) - \ln\left(\tan\frac{x}{2}+2\right)\right] + C$

$\frac{1}{t(t+2)} = \frac{1}{2t} - \frac{1}{2(t+2)}$

$= \frac{1}{2}\ln\left[\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+2}\right] + C.$

(4)

b) i)

$$\frac{2x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$2x = A(x^2+1) + (Bx+C)(x+1)$$

substituting $x = -1$

$$-2 = 2A$$

$$A = -1$$

equating coefficients of x^2, $A + B = 0$

$$\therefore B = 1$$

equating constant terms, $A + C = 0$

$$\therefore C = 1$$

$$\therefore \frac{2x}{(x+1)(x^2+1)} = \frac{-1}{x+1} + \frac{x+1}{x^2+1}$$

(3)

ii) $$\int_0^1 \frac{2x}{(x+1)(x^2+1)}\,dx$$

$$= \int_0^1 \left(\frac{-1}{x+1} + \frac{x+1}{x^2+1}\right) dx$$

$$= \int_0^1 \left(\frac{-1}{x+1} + \frac{1}{2}\cdot\frac{2x}{x^2+1} + \frac{1}{x^2+1}\right) dx$$

$$= \left[-\ln(x+1) + \frac{1}{2}\ln(x^2+1) + \tan^{-1}x\right]_0^1$$

$$= \left[\ln\frac{\sqrt{x^2+1}}{x+1} + \tan^{-1}x\right]_0^1$$

$$= -\frac{1}{2}\ln 2 + \frac{\pi}{4} - \ln 1 - 0$$

$$= \frac{\pi}{4} - \frac{\ln 2}{2}$$

(4)

c) $I_n = \int_0^1 e^x (2-x)^n \, dx$

i) $\int_0^1 e^x (2-x)^n \, dx$

$u = (2-x)^n \qquad v = e$

$u' = -n(2-x)^{n-1} \qquad v' = e$

$= uv - \int vu' \, dx$

$= \left[e^x (2-x)^n\right]_0^1 + n\int_0^1 e^x (2-x)^{n-1} \, dx$

$= e^x (2-1)^n - e^0 (2-0)^n + n\int_0^1 e^x (2-x)^{n-1} \, dx$

$I_n = e - 2^n + nI_{n-1}$ (3)

ii) Need to find I_4

$\therefore I_4 = e - 2^4 + 4I_3$

$I_3 = e - 2^3 + 3I_2$

$I_2 = e - 2^2 + 2I_1$

$I_1 = e - 2^1 + I_0$

$I_0 = \int_0^1 e \, dx$

$= \left[e^x\right]_0^1$

$= e - 1$

$\therefore I_1 = e - 2 + e - 1$

$= 2e - 3$

$I_2 = e - 4 + 2(2e-3)$

$= 5e - 10$

$I_3 = e - 2^3 + 3(5e-10)$

$= 16e - 38$

$I_4 = e - 2^4 + 4(16e-38)$

$= 65e - 168$

$\therefore \int_0^1 e(2-x)^4 \, dx$

$= 65e - 168$

(3)

Q.(10) Feedback:

(a) Just about everyone got to $\int \frac{1}{t^2+2t}\,.dt$

& most realised to use partial fractions from there. There was another interesting (& correct solution)

$$\int \frac{1}{t^2+2t}\,.dt$$

$$= -\frac{1}{2}\int \frac{-\frac{2}{t^2}}{1+\frac{2}{t}}\,.dt$$

$$= -\frac{1}{2}\ln\left[1+\frac{2}{t}\right]+C$$

[by $\int \frac{f'(x)}{f(x)}\,.dx = \ln(f(x))$.

$$= \frac{1}{2}\ln\left(\frac{t}{t+2}\right)+C$$

etc.

A really bad error was seen when some students though $\int \frac{1}{t^2+2t}\,.dt$ was an inverse tan.....?!!!

(b) Generally very well done. Just some silly mistakes.

(c)(ii) By FAR the most accurate method was by "Building Up"

$$I_0 = \int_0^1 e^x.dx = e-1$$

$$I_1 = e-2^1+1I_0$$

$$= 2e-3$$

$$I_2 = e-2^2+2I_1$$

etc.

Many students "built down" first THEN "built up" which was also good.

Students who tried to find I_4 in one line generally made silly mistakes.

TWO OTHER PROBLEMS:

(1) Some students forgot this was a DEFINITE integral & had e^x in their answer.

(2) v' does NOT equal dv!

$\int u.dv$ is actually $\int u.\frac{dv}{dx}.dx$. WASN'T PENALISED but ANNOYING!

Question 11

a) Let $I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x}\, dx$ —— (1)

replacing x by $\frac{\pi}{2} - x$

$$I = \int_0^{\pi/2} \frac{\sin^3\left(\frac{\pi}{2} - x\right)}{\sin^3\left(\frac{\pi}{2} - x\right) + \cos^3\left(\frac{\pi}{2} - x\right)}\, dx$$

$$I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x}\, dx \text{ —— (2)} \quad \left| \text{since } \sin\left(\frac{\pi}{2} - x\right) = \cos x, \ \cos\left(\frac{\pi}{2} - x\right) = \sin x \right.$$

Adding (1) & (2)

$$2I = \int_0^{\pi/2} \frac{\sin^3 x + \cos^3 x}{\sin^3 x + \cos^3 x}$$

$$= \int_0^{\pi/2} dx$$

$$= \frac{\pi}{4}$$

$$\therefore I = \frac{\pi}{4}$$

(4)

Reasonably well done.

b) i) $I_n = \int_0^{\pi/4} \sec^n x \, dx \, dx$

$= \int_0^{\pi/4} \sec^{n-2} x \, dx \sec^2 x \, dx$

$= uv - \int v u' dx$

$u = \sec^{n-2} x$

$u' = (n-2)\sec^{n-3} x \sec x \tan x$

$v = \tan x$

$v' = \sec^2 x$

$= \left[\tan x \sec^{n-2} x\right]_0^{\pi/4} - (n-2)\int_0^{\pi/4} \tan^2 x \sec^{n-2} dx$

$= \sqrt{2}^{n-2} - (n-2)\int_0^{\pi/4} (\sec^2 x - 1)(\sec^{n-2}) dx$

$= \sqrt{2}^{n-2} - (n-2)\left(\int_0^{\pi/4} \sec^n x \, dx - \int_0^{\pi/4} \sec^{n-2} dx\right)$

$= \sqrt{2}^{n-2} - (n-2) I_n + (n-2) I_{n-2}$

$\therefore I_n + (n-2) I_n = \sqrt{2}^{n-2} + (n-2) I_{n-2}$

$n I_n - I_n = \sqrt{2}^{n-2} + (n-2) I_{n-2}$

$I_n (n-1) = \sqrt{2}^{n-2} + (n-2) I_{n-2}$

$I_n = \frac{\sqrt{2}^{n-2}}{n-1} + \frac{n-2}{n-1} I_{n-2}$ (4)

ii) $I_6 = \int_0^{\pi/4} \sec^6 x \, dx$

$I_6 = \frac{\sqrt{2}^4}{5} + \frac{4}{5} I_4 = \frac{4}{5} + \frac{4}{5} I_4$

$I_4 = \frac{\sqrt{2}^2}{3} + \frac{2}{3} I_2 = \frac{2}{3} + \frac{2}{3} I_2$

$$I_2 = \int_0^{\pi/4} \sec^2 x \, dx$$

$$= \left[\tan x \right]_0^{\pi/4}$$

$$= 1$$

$$I_4 = \frac{2}{3} + \frac{2}{3} I_2$$

$$= \frac{4}{3}$$

$$I_6 = \frac{4}{5} + \frac{4}{5} I_4$$

$$= \frac{4}{5} + \frac{16}{15}$$

$$= \frac{28}{15}$$

(4)

(b) and (c)

- students who tried to condense everything into one or two steps made more mistakes.